D117: 6.3 Leontief Input-Output Problems

Economies with n Goods

Consider an economy with n different goods, which we refer to as "good 1," "good 2,"...,"good n." We assume that some quantity of all the goods in the economy gets used up during the production process of any given good. In a Leontief Input-Output problem, the goal is to figure out how much of each good we should input into the production process in order to meet a given external demand for those goods. Suppose we have an external demand for d_1 units of good 1, d_2 units of good 2, ..., d_n units of good n. Let $x_1, \ldots, x_n$ denote the number of units of good 1,..., good n, respectively, that must be inputted to meet this external demand. Written in matrix form, the problem is to solve the following matrix equation for X:

$$(I - A)X = D,$$

where

- A represents the "technology matrix" (of dimension $n \times n$). The (i, j) entry of A is equal to the number of units of good i used in the production of one unit of good j.
- D is the "external demand vector" (of dimension $n \times 1$):

$$D = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

- X is the "production schedule" (of dimension $n \times 1$):

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- I represents the usual identity matrix (of dimension $n \times n$).

You will have two types of questions:

1. A and X are given and you have to find the demand vector D. To find D you use the equation

$$D = (I - A)X.$$

2. A and D are given and you have to find the production schedule X. To find X you use the equation

$$X = (I - A)^{-1}D.$$

Example 1

Find the production schedule for the technology matrix and demand vector given below:

$$A = \begin{pmatrix} 0.3 & 0.1 \\ 0.4 & 0.8 \end{pmatrix}, \quad D = \begin{pmatrix} 20 \\ 50 \end{pmatrix}.$$

We have

$$(I - A)X = D \Longrightarrow X = (I - A)^{-1}D.$$

1. First we compute $(I-A)^{-1}$.
$$(I-A)=\begin{pmatrix}0.7 & -0.1\\ -0.4 & 0.2\end{pmatrix},$$
therefore
$$(I-A)^{-1}=\begin{pmatrix}2 & 1\\ 4 & 7\end{pmatrix}.$$

2. Now we can compute X:
$$X=(I-A)^{-1}D=\begin{pmatrix}2 & 1\\ 4 & 7\end{pmatrix}\begin{pmatrix}20\\ 50\end{pmatrix}=\begin{pmatrix}90\\ 430\end{pmatrix}$$

Example 2
Find the demand vector for the technology matrix and production schedule given below:
$$A=\begin{pmatrix}0.1 & 0.5\\ 0.7 & 0.5\end{pmatrix},\ X=\begin{pmatrix}135\\ 215\end{pmatrix}.$$

We have
$$(I-A)X=D\Longrightarrow X=(I-A)^{-1}D.$$

1. First we compute $(I-A)$:
$$(I-A)=\begin{pmatrix}0.9 & -0.5\\ -0.7 & 0.5\end{pmatrix}.$$

2. Now we can compute D:
$$D=(I-A)X=\begin{pmatrix}0.9 & -0.5\\ -0.7 & 0.5\end{pmatrix}\begin{pmatrix}135\\ 215\end{pmatrix}=\begin{pmatrix}14\\ 13\end{pmatrix}$$