**D117: 8.3 Cool stuff about Markov Chain sequel.**

**State/Probability Vector**
Consider a Markov chain with $N$ states. A state/probability vector vector for this Markov chain is

$$X = [x_1 \ \ x_2 \ \ \cdots \ \ x_N],$$

with

$$\begin{cases} x_1 + x_2 + \cdots + x_N = 1 \\ x_i \le 1, \ \ 1 \le i \le N \\ x_i = \text{probability that the system is in state } i. \end{cases}$$

**Example 1**
If we have a Markov chain with 3 states.

$$X = [0,1 \ \ 0.4 \ \ 0.5] \text{ is a state vector}$$

$$X = [0,1 \ \ 0.9 \ \ 0.3] \text{ is a NOT a state vector}$$

**State Vector after $k$ transition**
The state vector after $k$ transition is

$$X_k = XP^k \ \text{ OR } \ X_k = X_{k-1}P$$

**Example 2**
We have a Markov chain with the following transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.8 & 0.2 \end{bmatrix}$$

If the initial state vector start in state 2, what is the state vector after two transitions?

Since the system start in state 2, the initial state vector is $X_0 = [0 \ \ 1]$, and

$$X_2 = X_0 P^2 = [.56 \ \ .44].$$

**Regular Markov Chain**
Let $P$ be the transition matrix, $P$ is regular if there exists an integer $k$ such that $P^k$ has all positive entries.

**Example 3**
Let

$$P_1 = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}, \ \ P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We first compute $P_1^2$

$$P_1^2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix},$$

therefore $P_1$ is regular.

Now we compute $P_2^2$

$$P_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

so we compute $P_2^3$,

$$P_2^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_2,$$

We can conclude that

$$P_2^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ if k is even}$$

$$P_2^k = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ if k is odd}$$

therefore there is no such $k$ for which $P_2^k$ has all positive entries, thus $P_2$ is NOT regular.

**Stable probability**
let $P$ be the transition matrix for a regular Markov chain. There is a unique probability vector $W = [w_1 \ w_2 \ \cdots \ w_N]$ such that

$$WP = W.$$

Therefore to find $W$ you solve

$$W(P - I) = 0$$

**Example 4**
Let,

$$P = \begin{bmatrix} .25 & .75 \\ .6 & .4 \end{bmatrix}.$$

We want to find the vector $W$ of stable probabilities:

$$W(P - I) = 0.$$

This equivalent to

$$[w_1 \ w_2] \begin{bmatrix} -.75 & .75 \\ .6 & -.6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this gives

$$\begin{cases} -.75w_1 + .6w_2 = 0 \\ .75w_1 - .6w_2 = 0 \end{cases}$$

But we also have the fact that $W$ is a probability vector ($w_1 + w_2 = 1$), so we want to solve

$$\begin{cases} w_1 + w_2 = 1 \\ -.75w_1 + .6w_2 = 0 \\ .75w_1 - .6w_2 = 0 \end{cases}$$

The solution is $w_1 = 4/9$ and $w_2 = 5/9$,

$$W = [4/9 \ \ 5/9].$$