M251 Notes on Chapter 1

Classification of Differential Equations

- ODE: the unknown depends on one variable. linear or nonlinear
- linear ODE: it can be written as follows

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x). \tag{1}$$

- nonlinear ODE: when it is not linear.
- Order: highest order of derivatives.

Explicit/Implicit Solutions

- **Implicit Solution of an ODE:** A relation $G(x, y) = 0$ is said to be an implicit solution of an ordinary differential equation on an interval I, provided that there exists at least one function y that satisfies the relation as well as the differential equation on I.
- **Explicit Solution:** when you can write $y(x) = \ldots$

Initial-Value Problems

Solve

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}$$

Subject to

$$y(x_0) = y_0,\ \ y'(x_0) = y_1, \ldots,\ \ y^{(n-1)}(x_0) = y_{n-1}$$

How to find the interval of definition I

Also called interval of existence, interval of validity, or domain of the solution.

1. The ODE must be defined on the interval
2. The solution and its n derivatives must exist and be continuous on the interval
3. The solution must satisfy the ODE for all x in the interval
4. If there is an initial condition, the x value must be contained in the interval

Existence and Uniqueness for 1st order ODE

Theorem (1st order linear ODE): let

$$\begin{cases} y' + p(x)y = g(x), \\ y(x_0) = y_0. \end{cases} \tag{2}$$

If the functions $p(x)$ and $g(x)$ are continuous on an open interval (α, β) containing x_0, then there exists a unique solution of the IVP defined on (α, β).

Theorem (1st order ODE): let

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0. \end{cases} \tag{3}$$

If the functions $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangle $(x, y) \in (\alpha, \beta) \times (\gamma, \delta)$ containing (x_0, y_0). Then, in some interval $x \in (x_0 - h, x_0 + h) \subset (\alpha, \beta)$ there exists a unique solution $y(x)$ of the IVP.