M250 Notes on Laplace Transform

Laplace Transform

Definition. Given a function $f(t)$, $t \geq 0$, its Laplace transform is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f(t)dt.$$

Common functions

$$\mathcal{L}\{1\} = \frac{1}{s} \quad (s > 0) \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > a)$$
$$\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \quad (s > 0) \qquad \mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \quad (s > 0)$$

Properties of Laplace Transform

Linearity

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}$$

Derivative

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$
$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$
$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

The derivative of Laplace transforms

$$\mathcal{L}\{-tf(t)\} = F'(s), \text{ where } F(s) = \mathcal{L}\{f(t)\}$$

This also implies

$$\mathcal{L}\{tf(t)\} = -F'(s)$$
$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s) \text{ or, equivalently } \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Shift

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a), \text{ where } F(s) = \mathcal{L}\left\{f(t)\right\}$$

More functions

$$\begin{aligned}
\mathcal{L}\left\{t^n\right\} &= \frac{n!}{s^{n+1}} \quad (s>0) & \mathcal{L}\left\{e^{at}t^n\right\} &= \frac{n!}{(s-a)^{n+1}} \quad (s>a)\\
\mathcal{L}\left\{e^{at}\cos(bt)\right\} &= \frac{s-a}{(s-a)^2+b^2} \quad (s>a) & \mathcal{L}\left\{t\cos(bt)\right\} &= \frac{s^2-b^2}{(s^2+b^2)^2}\\
\mathcal{L}\left\{e^{at}\sin(bt)\right\} &= \frac{b}{(s-a)^2+b^2} \quad (s>a) & \mathcal{L}\left\{t\sin(bt)\right\} &= \frac{2bs}{(s^2+b^2)^2}
\end{aligned}$$

Inverse Laplace transform

Definition.

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t), \text{ if } F(s) = \mathcal{L}\left\{f(t)\right\}.$$

Solving ODE with Laplace Transform

1. Take Laplace transform on both sides of the ODE. You will get an equation for $Y(s)$, where $Y(s) = \mathcal{L}\left\{y(t)\right\}$.
2. Solve for $Y(s)$.
3. Take inverse transform to get $y(t) = \mathcal{L}^{-1}\left\{Y\right\}$.