M250 Notes on Section 10.1 and 10.2

Definition

When you solve a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ then your solution is

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The phase plane is plotting the solutions $(x_1(t), x_2(t))$ as $x_2 = f(x_1)$.

Equilibrium

A fixed pointca is $\mathbf{x}$ such that $\mathbf{x}' = \mathbf{0} \implies \mathbf{A}\mathbf{x} = \mathbf{0}$.

When $\mathbf{A}$ is invertible (i.e. $\det(\mathbf{A}) \neq 0$), the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has one fixed point $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Classification of Equilibrium at (0,0)

- Distinct real eigenvalues: $\lambda_1 \neq \lambda_2$
  - $\lambda_1 > 0$, $\lambda_2 > 0 \implies$ **source point, unstable**
  - $\lambda_1 \cdot \lambda_2 < 0 \implies$ **saddle, unstable**
  - $\lambda_1 < 0$, $\lambda_2 < 0 \implies$**sink node, asymptotically stable**
- Complex conjugate eigenvalues: $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.
  - $\alpha = 0 \implies$ **center, stable (but not asymptotically)**
  - $\alpha < 0 \implies$ **spiral point, asymptotically stable**
  - $\alpha > 0 \implies$ **spiral point, unstable**
- Repeated real eigenvalue: $\lambda = \lambda_1 = \lambda_2$.
  If you have two eigenvectors
  - $\lambda > 0 \implies$ **proper node, unstable**
  - $\lambda < 0 \implies$ **proper node, asymptotically stable**

  otherwise
  - $\lambda > 0 \implies$ **improper node, unstable**
  - $\lambda < 0 \implies$ **improper node, asymptotically stable**

Phase plane

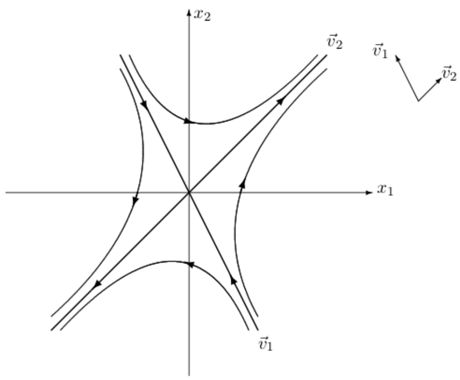
The system is

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

Distinct real eigenvalues: $\lambda_1 \neq \lambda_2$, then the general solution is

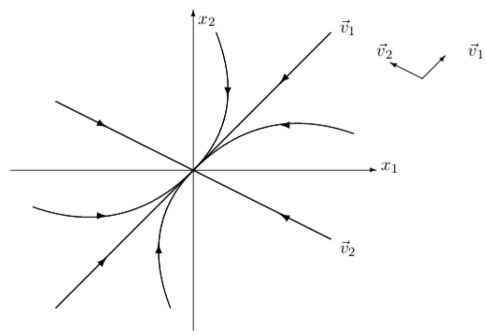
$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

$\lambda_1 \cdot \lambda_2 < 0$



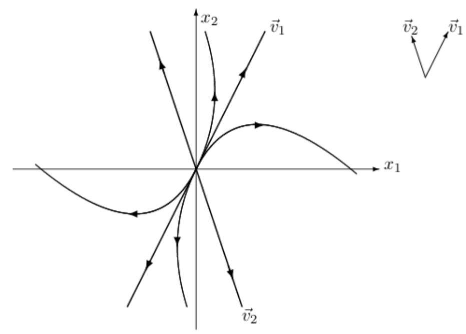
$(0,0)$ is a **saddle point**
$\Longrightarrow$ **unstable**

$\lambda_1 < 0,\ \lambda_2 < 0$



$(0,0)$ is a **sink node**
$\Longrightarrow$ **asymptotically stable**

$\lambda_1 > 0,\ \lambda_2 > 0$



$(0,0)$ is a **source node**
$\Longrightarrow$ **unstable**

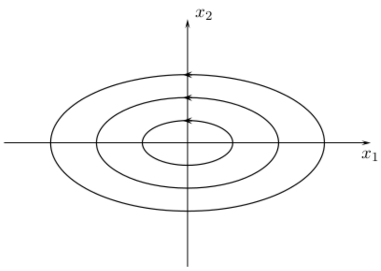
Complex conjugate eigenvalues: $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. Then the eigenvectors are $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$, and the general solution is

$$\mathbf{x} = C_1 e^{\alpha t}\left[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}\right] + C_2 e^{\alpha t}\left[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}\right]$$

Trick to remember

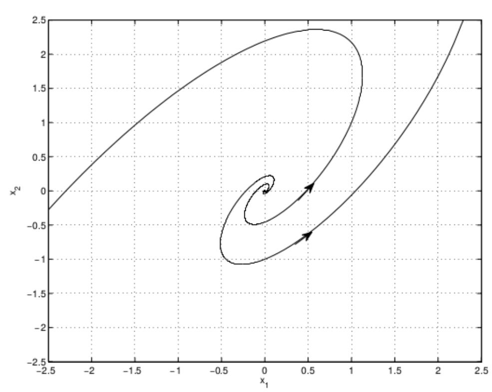
$$\mathbf{x} = C_1 \cdot \mathrm{Re}(e^{\lambda_1 t} v_1) + C_2 \cdot \mathrm{Im}(e^{\lambda_1 t} v_1)$$

$\alpha = 0$



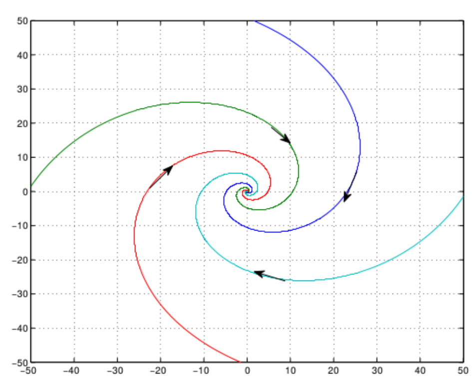
$(0,0)$ is a **center point**
$\Longrightarrow$ **stable**

$\alpha > 0$



$(0,0)$ is a **spiral point**
$\Longrightarrow$ **unstable**

$\alpha < 0$



$(0,0)$ is **spiral point**
$\Longrightarrow$ **asymptotically stable**

Repeated real eigenvalue: $\lambda = \lambda_1 = \lambda_2$.

If you can find two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{x} = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2$$

If not, take $\mathbf{v}$ eigenvector associated to λ

$$\mathbf{x} = C_1 e^{\lambda t}\mathbf{v} + C_2\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\boldsymbol{\eta}\right)$$

where $\boldsymbol{\eta}$ solves $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$

If there are two linearly independent eigenvector	
$\lambda < 0$ $(0, 0)$ is an **proper node** $\implies$ **asymptotically stable**	$\lambda > 0$ $(0, 0)$ is an **proper node** $\implies$ **unstable**

If there is only one linearly independent eigenvector	
$\lambda < 0$ $(0, 0)$ is an **improper node** $\implies$ **asymptotically stable**	$\lambda > 0$ $(0, 0)$ is an **improper node** $\implies$ **unstable**