M250 First Order ODEs

Solving a Separable Equation

Let

$$\frac{dy}{dx} = g(x)h(y).$$

Then the implicit solution is given by

$$\int \frac{1}{h(y)} dy = \int g(x) dx.$$

To obtain the explicit solution you need to solve for y.

⚠ Don't forget the interval of validity.

Example:

$$\frac{dy}{dx} = -\frac{x}{y}, \;\; y(4) = -3$$

1. We rewrite the equation as

$$ydy = -xdx \Longrightarrow \int ydy = -\int xdx \Longrightarrow \frac{y^2}{2} = -\frac{x^2}{2} + C$$

2. The implicit solution is

$$y^2 + x^2 = C$$

3. The general solution is

$$y = \pm\sqrt{C - x^2}, \;\; x \in [-\sqrt{C}, \sqrt{C}]$$

4. Now we check the initial condition $y(4) = -3$. Since $y(4)$ is negative we have to choose the negative solution

$$y = -\sqrt{C - x^2} \Longrightarrow -3 = -\sqrt{C - 4^2} \Longrightarrow C = 25$$

The specific solution is

$$\boxed{y = -\sqrt{25 - x^2}, \;\; x \in (-5, 5)}$$

Solving a 1st order linear ODE

Let

$$y' + P(x)y = f(x).$$

Then the general solution is

$$y(x) = \frac{1}{\mu(x)} \left(\int \mu(s)g(s)ds \right),$$

and μ, called the integrating factor, is

$$\mu(x) = \exp\left(\int P(x)dx \right).$$

⚠ Don't forget the interval of validity.

Example:

$$x\frac{dy}{dx} + y = x^2, \;\; y(1) = 4$$

1. We rewrite the equation as

   $$x\frac{dy}{dx} + xy = x^2 \Longrightarrow \frac{dy}{dx} + \frac{y}{x} = x$$

   Since we have $1/x$ in the ODE, the interval will have to be $(-\infty, 0)$ or $(0, \infty)$. But from the initial condition we pick $(0, \infty)$.

2. The integrating factor is

   $$\mu(x) = \exp\left(\int \frac{1}{x}dx \right) = e^{\ln(x)} = x$$

   (Remark: we omit the absolute value around x since $x > 0$)

3. The general solution is

   $$y = \frac{1}{x}\int s^2 ds = \frac{x^2}{3} + \frac{C}{x}, \text{ for } x > 0$$

4. Now we check the initial condition $y(1) = 4$.

   $$4 = \frac{1}{3} + C \Longrightarrow C = \frac{11}{3}$$

   The specific solution is

   $$\boxed{y(x) = \frac{x^2}{3} + \frac{11}{3x}, \;\; x \in (0, +\infty)}$$