M250 Preliminary results on ODE

N^{th} Order ODE

Linear ODE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x).$$

Homogeneous linear ODE

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0.$$

Initial-Value Problem (IVP)

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = g(x).$$

with

$$y(x_0) = y_0, \;\; y'(x_0) = y_1, \;\; \ldots, \;\; y^{(n-1)}(x_0) = y_{n-1}$$

and $y_0, \ldots, y_{n-1}$ are real constants.

Existence and Uniqueness for an IVP problem

If

- the coefficients $a_n(x), \ldots, a_0(x)$ are continuous on an interval I
- $a_n(x) \neq 0 \; \forall x \in I$
- the initial value $x_0 \in I$

then there exists a unique solution $y(x)$ of the IVP.

Superposition principal - homogeneous equation

If $y_1, y_2, \ldots, y_k$ are solution of an linear homogeneous ODE then linear combination

$$y = C_1 y_x(x) + C_2 y_2(x) + \cdots + C_k y_k(x),$$

where $C_1, C_2, \ldots, C_k \in \mathbb{R}$, is also also a solution of the linear homogeneous ODE.

Second Order ODE

Second order Initial-Value Problem (IVP)

$$a_2(x)y''(x) + a_1(x)y' + a_0(x)y = g(x).$$

with

$$y(x_0) = y_0, \;\; y'(x_0) = y_1$$

Second Order Boundary Value Problem (BVP)

$$a_2(x)y''(x) + a_1(x)y' + a_0(x)y = g(x).$$

with ONE of the following pair

$$y(a) = y_0, \;\; y(b) = y_1$$

$$y'(a) = y_0, \;\; y(b) = y_1$$

$$y(a) = y_0, \;\; y'(b) = y_1$$

$$y'(a) = y_0, \;\; y'(b) = y_1$$

Superposition principal

If $y_1(t)$ and $y_2(t)$ are solution of a second order homogeneous ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0, \tag{1}$$

Then $\forall C_1, C_2 \in \mathbb{R}$, $y = C_1y_1 + C_2y_2$ is also a solution.

Existence & Uniqueness for second order IVP

Let us define the following second order linear ODE

$$\begin{cases} y'' + p(x)y' + q(x)y = g(x), \\ y(x_0) = y_0, \\ y'(x_0) = y_1. \end{cases} \tag{2}$$

If the functions p, q, and g are continuous on the interval I, containing the point x_0. Then there exists a unique solution of the ODE that is defined on I.

Wronskian

Let $f(x)$ and $g(x)$ be two functions. Then

$$W(f,g)(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

- $W(f,g) \equiv 0 \implies f$ and g are linearly dependent.
- Otherwise $\implies f$ and g are linearly independent.

Fundamental Set of Solutions

Any set $y1, y2$ of two linearly independent solutions of an homogeneous linear 2nd-order differential equation on an interval I is said to be a fundamental set of solutions on the interval.

This means that any solution $y(x)$ can be written as a linear combination of y_1 and y_2:

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

General Solutions

Consider the following equation

$$y'' + p(x)y' + q(x)y = 0, \tag{3}$$

If y_1 and y_2 are both solution and $W(y_1, y_2)(x) \neq 0$ then the general solution of the ODE is

$$y = C_1 y_1 + C_2 y_2$$