M250 Reduction of order

Theorems

Wronskian

Let $f(x)$ and $g(x)$ be two functions. Then

$$W(f,g)(x) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

- $W(f,g) \equiv 0 \implies f$ and g are linearly dependent.
- Otherwise $\implies f$ and g are linearly independent.

Abel's identity

Let y_1 and y_2 be solution of

$$y'' + p(x)y' + q(x)y = 0, \tag{1}$$

Then

$$W(y_1, y_2) = C \cdot \exp\left(-\int p(x)dx\right).$$

General Solutions

Consider the following equation

$$y'' + p(x)y' + q(x)y = 0,$$

If y_1 and y_2 are both solution and $W(y_1, y_2)(x) \neq 0$ then the general solution of the ODE is

$$y = C_1y_1 + C_2y_2$$

Find a second solution (reduction of order)

You have the equation

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

and assume you know one solution y_1. In order to find the general solution of this ODE you need to find a second independent solution. First, you define $\boxed{y_2 = v(x)y_1}$ and you obtain the following

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1(v' y_1 + v y_1') - v y_1 y_1' = v' y_1^2,$$

and from Abel's identity with $C = 1$ we have

$$W(y_1, y_2) = \exp\left(-\int p(x)dt\right).$$

Thus you solve for v

$$v' y_1^2 = \exp\left(-\int p(x)dt\right)$$

and you obtain your second solution $y_2 = v y_1$.

Example: The function $y_1 = x^2$ is solution of

$$x^2 y'' - 3xy' + 4y = 0.$$

We define $y_2 = v y_1$ and from Abel's theorem we have

$$v' x^4 = \exp\left(\int \frac{3}{x} dx\right) = x^3 \Longrightarrow v' = \frac{1}{x} \Longrightarrow v(x) = \ln(x).$$

Thus $y_2 = x^2 \ln(x)$ and the general solution of the ODE is

$$y(x) = C_1 x^2 + C_2 x^2 \ln(x)$$