M250 Notes on Non-homogeneous Second Order ODE

Method/theorem

The goal is to solve

$$y'' + p(x)y' + q(x)y = g(x) \tag{N}$$

Steps

1. Solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \tag{H}$$

   i.e finding y_1 and y_2, two linearly independent solutions. Then we define the general solution of the homogeneous system:

$$y_H = C_1 y_1 + C_2 y_2.$$

2. Find a **particular solution** y_P of the non-homogeneous equation (N) (using the method of undetermined coefficients)

3. The **general solution** of (N) is

$$y = y_H + y_P = C_1 y_1 + C_2 y_2 + y_P$$

Method of undetermined coefficients

Now, we only consider the following ODE with constant coefficients

$$\boxed{ay'' + by' + cy = g(x)}$$

Let r_1 and r_2 be the roots of the characteristic equation of the homogeneous problem.

Case 1: $g(x) = \gamma e^{\alpha x}$

case	form of the particular solution y_P
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$y_P = Ae^{\alpha x}$
$r_1 = \alpha$ and $r_2 \neq \alpha$	$y_P = Axe^{\alpha x}$
$r_1 = r_2 = \alpha$	$y_P = Ax^2 e^{\alpha x}$

Case 2: $g(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$

case	form of the particular solution y_P
$c \neq 0$	$y_P = A_n x^n + \cdots + A_1 x + A_0$
$c = 0$ and $b \neq 0$	$y_P = x\left(A_n x^n + \cdots + A_1 x + A_0\right)$
$c = b = 0$	$y_P = x^2\left(A_n x^n + \cdots + A_1 x + A_0\right)$

Case 3: $g(x) = \gamma \cos(\beta x) + \delta \sin(\beta x)$

case	form of the particular solution y_P
$r_{1,2} \neq \pm\beta i$	$y_P = A\cos(\beta x) + B\sin(\beta x)$
$r_{1,2} = \pm\beta i$	$y_P = x\left(A\cos(\beta x) + B\sin(\beta x)\right)$

Case 4: $g(x) = e^{\alpha x}\left[\alpha_n x^n + \cdots + \alpha_1 x + \alpha_0\right]$

case	form of the particular solution y_P
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$y_P = e^{\alpha x}\left[A_n x^n + \cdots + A_1 x + A_0\right]$
$r_1 = \alpha$ and $r_2 \neq \alpha$	$y_P = xe^{\alpha x}\left[A_n x^n + \cdots + A_1 x + A_0\right]$
$r_1 = \alpha$ and $r_2 = \alpha$	$y_P = x^2 e^{\alpha x}\left[A_n x^n + \cdots + A_1 x + A_0\right]$

Case 5: $g(x) = e^{\alpha x}\left[\gamma \cos(\beta x) + \delta \sin(\beta x)\right]$

case	form of the particular solution y_P
$r_{1,2} \neq \alpha \pm \beta i$	$y_P = e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$
$r_{1,2} = \alpha \pm \beta i$	$y_P = xe^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]$

Case 6: $g(x) = e^{\alpha x}\left[\gamma \cos(\beta x) + \delta \sin(\beta x)\right](\eta_n x^n + \cdots + \eta_1 x + \eta_0)$
or $= e^{\alpha x}\left[(\eta_n x^n + \cdots + \eta_1 x + \eta_0)\cos(\beta x) + (\mu_n x^n + \cdots + \mu_1 x + \mu_0)\sin(\beta x)\right]$

case	form of the particular solution y_P
$r_{1,2} \neq \alpha \pm \beta i$	$y_P = e^{\alpha x}\left[(A_n x^n + \cdots + A_1 x + A_0)\cos(\beta x) + (B_n x^n + \cdots + B_1 x + B_0)\sin(\beta x)\right]$
$r_{1,2} = \alpha \pm \beta i$	$y_P = xe^{\alpha x}\left[(A_n x^n + \cdots + A_1 x + A_0)\cos(\beta x) + (B_n x^n + \cdots + B_1 x + B_0)\sin(\beta x)\right]$

Case 7: $g(x)$ is a sum of the previous cases, i.e.

$$g(x) = g_1(x) + \cdots + g_n(x).$$

You find a particular solution y_{P_i} for each $g_i(x)$ term as if it were the only term in $g(x)$. Then

$$y_P = y_{P_1}(x) + \cdots + y_{P_n}(x).$$

Rules

1. Choose y_P as the same form as $g(x)$.
2. If $g(x)$ is a solution to the homogeneous equation then multiply by x.
3. If $xg(x)$ is still solution of the homogeneous equation then multiply by x^2