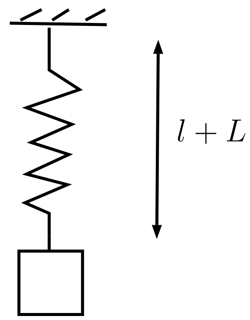
M250 Notes on Spring Mass System

Mass Spring System

$$mu'' + \gamma u' + ku = F(t)$$

where $m > 0$ and $k > 0$.

A mass spring system represents a mass m suspended at the end of a spring:



$$mu'' = ma = \sum f = mg - k(L + u) - \gamma u' + F$$

Since $mg = kL$ we have $mu'' + \gamma u' + ku = F(t)$ with

- $u(t)$ = displacement of the mass relative to its equilibrium position.
- m = mass ($m > 0$)
- γ = damping constant ($\gamma \geq 0$)
- k = spring (Hooke's) constant ($k > 0$)
- g = gravitational constant
- L = elongation of the spring caused by the weight
- $F(t)$ = Externally applied forcing function, if any
- $u(t_0)$ = initial displacement of the mass
- $u(t_0)$ = initial velocity of the mass

Unforced Mass-Spring System ($F(t) = 0$)

Undamped free vibration ($\gamma = 0$)

$$mu'' + ku = 0$$

with $m > 0$ and $k > 0$. When we solve it, we obtain

$$mr^2 + k = 0, \;\; r^2 = -\frac{k}{m}, \;\; r_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0, \;\text{where } w_0 = \sqrt{\frac{k}{m}}$$

The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

or

$$u(t) = A\cos(\omega_0 t - \varphi) = Asin(\omega_0 t + \varphi)$$

with

- **Frequency**: $\omega_0 = \sqrt{\frac{k}{m}}$
- **Period**: $T = \frac{2\pi}{\omega_0}$
- **Amplitude**: $R = \sqrt{C_1^2 + C_2^2}$
- **Phase**: $\varphi = \arctan\dfrac{C_1}{C_2}$.

Damped free vibration ($\gamma \neq 0$)

$$mu'' + \gamma u' + ku = 0$$

with $m > 0$, $\gamma > 0$, and $k > 0$. Let $\Delta = \gamma^2 - 4km$, the roots are

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$$

- if $\Delta > 0$ then **Overdamped** $\Longrightarrow u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- it $\Delta = 0$ then **Critically damped** $\Longrightarrow u(t) = C_1 e^{rt} + C_2 t e^{rt}$.
- it $\Delta < 0$ then **Underdamped**. $\Longrightarrow u(t) = e^{-\lambda t}\left[C_1 \cos(\mu t) + C_2 \sin(\mu t)\right]$.

Undamped with forced vibration

Undapend with a force

$$mu'' + ku = F_0 \cos(\omega t) \text{ or } mu'' + ku = F_0 \sin(\omega t),$$

with $m > 0$ and $k > 0$.

Case 1 $\omega \neq \omega_0$
The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + A \cos(\omega t) + B \sin(\omega t).$$

Case 2 $\omega = \omega_0$
The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + At \cos(\omega_0 t) + Bt \sin(\omega_0 t).$$

We say that the system is in **resonance**.

$$\boxed{\omega = \omega_0 \Longrightarrow \text{Resonance}}$$

Example 1. A mass weighing 10 lb stretches a spring 2 in. We neglect damping. If the mass is displaced an additional 2 in, and is then set in motion with initial upward velocity of 1 ft/sec, determine the position, frequency, period, amplitude and phase of the motion

Example 2. A mass weighing 2 pounds stretches a spring 6 inches. At $t > 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of 4/3 ft/s . Determine the equation of motion.

Example 3. A mass of 1 kg is hanging on a spring with k = 1. The mass is in a medium that exerts a viscous resistance of 6 newton when the mass has a velocity of 48 m/sec. The mass is then further stretched for another 2m, then released from rest. Find the position u(t) of the mass

Example 4. A mass weighing 8 pounds stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the mass is initially released from the equilibrium position with an upward velocity of 3 ft/s.