M251 Notes on Exact Equations

Partial derivative:

$$\frac{df}{dt}(x(t), y(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$
$$\frac{df}{dx}(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

Notation

$$\phi_x = \frac{\partial \phi}{\partial x}(x, y)$$
$$\phi_y = \frac{\partial \phi}{\partial y}(x, y)$$

Exact equation

Let us solve the following ODE:

$$M(x, y) + N(x, y) \cdot y' = 0$$

The goal is to find $y(x)$ that is solution. We want to write it as $\phi(x, y)$ where $y = y(x)$.

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{dy}{dx} = 0 \Longrightarrow \frac{d\phi}{dx} = 0 \Longrightarrow \phi(x, y) = C$$

1. Is the ODE is exact? ($\phi_{xy} = \phi_{yx}$)

   $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

   This means that ϕ exists.

2. Find ϕ such that

   $$\frac{\partial \phi}{\partial x} = M(x, y) \text{ and } \frac{\partial \phi}{\partial y} = N(x, y)$$

3. The general solution is $\phi(x, y) = C$, find C with initial condition.

Example: $(3x^2 + y) + (x + 3y^2)y' = 0$ and $y(1) = 1$

1. $M_y = 1$ and $N_x = 1$ therefore it is exact.
2. $\phi_x = 3x^2 + y$

$$\Longrightarrow \phi = \int (3x^2 + y)dx + h(y) \Longrightarrow \phi = x^3 + yx + h(y)$$

   $\phi_y = x + 3y^2$

$$\Longrightarrow x + h'(y) = x + 3y^2 \Longrightarrow h'(y) = 3y^2 \Longrightarrow h(y) = y^3$$

3. The general solution ($\phi = C$) is

$$x^3 + y^3 + yx = C$$

   $y(1) = 1 \Longrightarrow C = 3$, so the solution of the IVP is

$$x^3 + y^3 + yx = 3$$