M251 Notes on Fourier Series

Fourier Series

Let $f(x)$ be a $2L$-periodic function then its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

with

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) dx,$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx,$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx,$$

Odd/Even Function

- If $f(x)$ is an **even** $2L$-periodic function, it has **Fourier cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi x}{L}$$

with

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad n = 0, 1, 2, \ldots$$

- If $f(x)$ is an **odd** $2L$-periodic function, it has **Fourier sine series**

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin \frac{n\pi x}{L}$$

with

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \ldots$$

Half-range expansion

If a function $f(x)$ is only defined on an interval $[0, L]$, we can extend/expand the domain into the whole real line by periodic expansion. There are two ways of doing this

- **Even expansion**.

$$\tilde{f}(x) = \begin{cases} f(-x), & \text{if } -L < x < 0 \\ f(x), & \text{if } 0 \leq x \leq L \end{cases} \qquad \text{and } \tilde{f}(x + 2L) = \tilde{f}(x).$$

- **Odd expansion**.

$$\tilde{f}(x) = \begin{cases} -f(-x), & \text{if } -L < x < 0 \\ f(x), & \text{if } 0 \leq x \leq L \end{cases} \qquad \text{and } \tilde{f}(x + 2L) = \tilde{f}(x).$$

Properties of Fourier series

Linearity. If $f(x)$ and $g(x)$ are two $2L$-periodic functions with Fourier coefficients (a_n, b_n) and $(\bar{a}_n, \bar{b}_n)$ respectively then $(f + g)(x)$ has Fourier coefficients $(a_n + \bar{a}_n, b_n + \bar{b}_n)$.

Convergence. Let f be a $2L$-periodic function and assume that f is piecewise continuous, then

1. The Fourier series converges to $f(x)$ at all points x when f is continuous.
2. At a point x when f is discontinuous, the Fourier series converges to the mid point value, i.e

$$\frac{1}{2}\left[f(x^-) + f(x^+)\right]$$