M251 Notes on Higher Order Linear ODE

The general form of a linear equation of n-th order, with $y(t)$ as the unknown, is

$$\boxed{y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t).} \tag{A}$$

We would need to assign n initial conditions. Normally, the followings are given

$$y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0).$$

Existence and Uniqueness. If the coefficient functions $p_0(t)$, $p_1(t)$, ... , $p_{n-1}(t)$ are continuous (and bounded) on an open interval I containing t_0 , then the equation (A) has a unique solution on the interval I.

Homogeneous equations.

$$\boxed{y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0.} \tag{H}$$

There are n solutions, y_1 , y_2 , ... y_n , linearly independent, that forms a set of fundamental solutions, whose linear combination gives the general solution:

$$y_H(t) = C_1y_1 + C_2y_2 + \cdots + C_ny_n$$

where the constants C_1 , C_2 , ... , C_n are determined by the n initial conditions

Homogeneous equations with constant coefficients.

$$\boxed{a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0.} \tag{H}$$

Characteristic equation

$$a_nr_n + \cdots + a_1r + a_0 = 0$$

Root type	Solution(s)
r is real, un-repeated	e^{rt}
r is real, double root	e^{rt}, te^{rt}
r is real, triple root	e^{rt}, te^{rt}, t^2e^{rt}
r is real, repeated m times	e^{rt}, te^{rt},... ,t^me^{rt}
$r = \lambda \pm i\mu$, complex	$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$
$r = \lambda + i\mu$, double roots	$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$, $te^{\lambda t}\cos(\mu t)$, $te^{\lambda t}\sin(\mu t)$
$r = \lambda + i\mu$, triple roots	$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$, $te^{\lambda t}\cos(\mu t)$, $te^{\lambda t}\sin(\mu t)$, $t^2e^{\lambda t}\cos(\mu t)$, $t^2e^{\lambda t}\sin(\mu t)$
$r = \lambda + i\mu$, repeated m times	$e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$, $te^{\lambda t}\cos(\mu t)$, $te^{\lambda t}\sin(\mu t)$, $\ldots$, $t^me^{\lambda t}\cos(\mu t)$, $t^me^{\lambda t}\sin(\mu t)$