M251 Notes on Chapters 1 and 2

Classification of Differential Equations

- ODE: the unknown depends on one variable. linear or nonlinear
  - linear ODE: it can be written as follows
$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t). \tag{1}$$
  - nonlinear ODE: when it is not linear.
- PDE: the unknown depends on multiple variable.
- Order: highest order of derivatives.

Solving a 1st order linear ODE

It looks like this:
$$y' + p(t)y = g(t). \tag{2}$$
Then the solution is
$$\begin{aligned} \mu(t) &= \exp\left(\int p(t)dt\right) \text{(Integrating factor)} \\ y(t) &= \frac{1}{\mu(t)}\left(\int \mu(t)g(t)dt\right) \text{(General solution)} \end{aligned} \tag{3}$$
Don't forget the interval of validity.

Solving a 1st order ODE with separable variables

It looks like this:
$$y' = \frac{M(x)}{N(y)}. \tag{4}$$
Then the implicit solution is given by
$$\int N(y)dy = \int M(x)dx. \tag{5}$$
Then you solve for y to obtain the explicit solution. Don't forget the interval of validity.

Existence and Uniqueness for 1st order ODE

Theorem (1st oder linear ODE): let

$$\begin{cases} y' + p(t)y = g(t), \\ y(t_0) = y_0. \end{cases} \tag{6}$$

If the functions $p(t)$ and $g(t)$ are continuous on an open interval (α, β), then there exists a unique solution of the IVP defined on (α, β).

Theorem (1st oder ODE): let

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases} \tag{7}$$

If the functions $f(t, y)$ and $\partial f/\partial y$ are continuous on a rectangle $(t, y) \in (\alpha, \beta) \times (\gamma, \delta)$ containing (t_0, y_0). Then, in some interval $t \in (t_0 - h, t_0 + h) \subset (\alpha, \beta)$ there exists a unique solution $y(t)$ of the IVP.

Economy ODE

Investing money. Assume you start an account with A_0 dollars, that gains interest at a rate r per unit time compounded continuously. In addition you add k amount of money per unit time. Then the ODE is

$$\begin{cases} y' = ry + k, \\ y(0) = A_0, \end{cases} \tag{8}$$

where

$$y(t) = \text{the amount of money at time } t.$$

Borrow money. Assume you borrow A_0 dollars, at an interest rate of r per unit time compounded continuously, and you make a payment of k dollars per unit of time. Then the ODE is

$$\begin{cases} y' = ry - k, \\ y(0) = A_0, \end{cases} \tag{9}$$

where

$$y(t) = \text{the amount of money you owe } t.$$

You are done paying the loan at time t_f, when $y(t_f) = 0$.

Mixing solution

A tank contains Q_0 of solute dissolved in S_0 amount of solvent. Additional solvent at a concentration of c_i of solute flows into the tank at a rate r_i. Also, the mixed content is pumped out of the take at a rate r_o.

- $Q(t)$ = amount of solute in tank at time t.

$$\begin{cases} Q' = r_i c_i - r_o \dfrac{Q}{S(t)}, \\ Q(0) = Q_0. \end{cases} \tag{10}$$

- $S(t)$ = volume of solution in tank at time t.

$$S(t) = S_0 + (r_i - r_o)t, \tag{11}$$

Physics

Object falling $m \cdot a = \sum Forces$. The unknown is the velocity $dv(t)/dt = a(t)$. The forces are the gravitational force mg and the drag force $k|v|$ (against the direction of v).

$$mv' = mg - kv. \tag{12}$$

Let $h(t)$ be the height of the object at time t

$$h(t) = \int v(t)dt. \tag{13}$$

Exercises

Example 1. You put $\$A_0$ in a bank account with an interest of 8%, compounded continuously, find doubling time.

Example 2. A radio active material is reduced to 1/3 after 10 years. Find its half life.

Example 3. Start an IRA account at age 25. Suppose deposit \$2000 at the beginning and \$2000 each year after. Interest rate 8% annually, but assume compounded continuously. Find total amount after 40 years.

Example 4. A home-buyer can pay \$800 per month on mortgage payment. Interest rate is r annually, (but compounded continuously), mortgage term is 20 years. Determine maximum amount this buyer can afford to borrow. Calculate this amount for r = 5% and r = 9% and observe the difference.

Example 5. At $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of r gal/min. At the same time, the well-mixed mixture is draining from the tank at the same rate.

1. Find the amount of salt in the tank at any time $t \geq 0$.
2. When $t \to \infty$, meaning after a long time, what is the limit amount Q_L ?

Example 6. Tank contains 50 lb of salt dissolved in 100 gal of water. Tank capacity is 400 gal. From $t = 0$, 1/4 lb of salt/gal is entering at a rate of 4 gal/min, and the well-mixed mixture is drained at 2 gal/min. Find:

1. time t when it overflows;
2. amount of salt before overflow;
3. the concentration of salt at overflow.

Example 7. A ball with mass 0.5 kg is thrown upward with initial velocity 10 m/sec from the roof of a building 30 meter high. Assume air resistance is $|v|/20$.

1. Find the max height above ground the ball reaches.
2. Find the time when the ball hit the ground.
3. Find the total distance traveled by the ball when it hits the ground.

Example 8. A swimming pool initially contains 1000 m 3 of stale, unchlorinated water. Water containing 2 grams per m^3 of chlorine flows into the pool at a rate of 4 m^3 per minute. The well-mixed content of the pool is drained at the same rate. Find the time when the chlorine concentration in the pool reaches 1 gram per m^3.

Example 9.(Exam 1, summer 2002): A 400-liter tank is initially filled with 100 liters of dye solution with a dye concentration of 5 grams/liter. Pure water flows into the tank at a rate of 3 liters per minute. The well-stirred solution is drained at a rate of 2 liters per minute. Find the concentration of dye in the tank at the time that the tank is completely filled.

Example 10. A 100 kg Unmanned Aerial Vehicle (UAV) possesses propulsive force of 10000 N and has drag coefficient k = 4. Find the velocity function of its flight.

Example 11. (final exam, fall 2007): A college student borrows $5000 to buy a car. The lender charges interest at an annual rate of 10payments continuously at a constant annual rate k. Determine the payment rate k that is required to pay off the loan in 5 years.

Example 12. The present value of a lottery jackpot A lucky college student has won the lotterys ten million dollars jackpot. The winning is paid out equally over 20 years. Assume the payout is made continuously and the annual interest rate is constant 8% over the 20-year period. How much is the jackpot worth in todays dollar?