

M251 Notes on Laplace Transform

Laplace Transform

Definition. Given a function $f(t)$, $t \geq 0$, its Laplace transform is defined as

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = \lim_{A \rightarrow \infty} \int_0^A e^{-st} f(t) dt.$$

Common functions.

$$\begin{aligned} \mathcal{L}\{1\} &= \frac{1}{s} \quad (s > 0) & \mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \quad (s > a) \\ \mathcal{L}\{\cos(at)\} &= \frac{s}{s^2+a^2} \quad (s > 0) & \mathcal{L}\{\sin(at)\} &= \frac{a}{s^2+a^2} \quad (s > 0) \end{aligned}$$

Properties of Laplace Transform

Linearity.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

Derivative.

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

The derivative of Laplace transforms.

$$\mathcal{L}\{-tf(t)\} = F'(s), \text{ where } F(s) = \mathcal{L}\{f(t)\}$$

This also implies

$$\mathcal{L}\{tf(t)\} = -F'(s)$$

$$\mathcal{L}\{(-t)^n f(t)\} = F^{(n)}(s) \text{ or, equivalently } \mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$$

Shift.

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a), \text{ where } F(s) = \mathcal{L}\{f(t)\}$$

More functions.

$$\begin{aligned}\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \quad (s > 0) & \mathcal{L}\{e^{at}t^n\} &= \frac{n!}{(s-a)^{n+1}} \quad (s > a) \\ \mathcal{L}\{e^{at}\cos(bt)\} &= \frac{s-a}{(s-a)^2+b^2} \quad (s > a) & \mathcal{L}\{t\cos(bt)\} &= \frac{s^2-b^2}{(s^2+b^2)^2} \\ \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{b}{(s-a)^2+b^2} \quad (s > a) & \mathcal{L}\{t\sin(bt)\} &= \frac{2bs}{(s^2+b^2)^2}\end{aligned}$$

Inverse Laplace transform

Definition.

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \text{ if } F(s) = \mathcal{L}\{f(t)\}.$$

Solving ODE with Laplace Transform

1. Take Laplace transform on both sides of the ODE. You will get an equation for $Y(s)$, where $Y(s) = \mathcal{L}\{y(t)\}$.
2. Solve for $Y(s)$.
3. Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y\}$.