M251 Notes on PDEs

Heat equations

Homogeneous Dirichlet

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L, \;\; t > 0 \\ u(0,t) = 0, \;\; u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

The general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)}$$

The particular solution is found by computing the C_n's, where the C_n's are the Fourier sine coefficients for the odd periodic extension of $f(x)$.

Homogeneous Neumann

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L, \;\; t > 0 \\ u_x(0,t) = 0, \;\; u_x(L,t) = 0, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

The general solution is

$$\boxed{u(x,t) = C_0 + \sum_{n=1}^{\infty} C_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)}$$

The particular solution is found by computing the C_n's, where the C_n's are the Fourier cosine coefficients for the even periodic extension of $f(x)$.

Non-Homogeneous Dirichlet

$$\begin{cases} u_t = \alpha^2 u_{xx}, & 0 < x < L, \;\; t > 0 \\ u(0,t) = a, \;\; u(L,t) = b, & t > 0 \\ u(x,0) = f(x), & 0 < x < L \end{cases}$$

The steady state solution is

$$U(x) = a + \frac{b-a}{L}x$$

The general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{\alpha n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right) + U(x)}$$

The particular solution is found by computing the C_n using the initial condition.

Wave equations

The vertical displacement (u) of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$\begin{cases} u_{tt} = \alpha^2 u_{xx}, & 0 < x < L, \;\; t > 0 \\ u(0,t) = 0, \;\; u(L,t) = 0, & t > 0 \\ u(x,0) = f(x), \;\; u_t(x,0) = g(x), & 0 < x < L \end{cases}$$

The general solution is:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\left(\frac{\alpha n \pi}{L} t\right) + D_n \sin\left(\frac{\alpha n \pi}{L} t\right) \right] \sin\left(\frac{n\pi}{L} x\right)}$$

The particular solution is found by:

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx,$$

$$D_n = \frac{2}{\alpha n \pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx.$$