M251 Notes on Phase Plane

Definition

When you solve a system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ then your solution is

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The phase plane is plotting the solutions $(x_1(t), x_2(t))$ as $x_2 = f(x_1)$.

Equilibrium

A fixed pointca is $\mathbf{x}$ such that $\mathbf{x}' = \mathbf{0} \Longrightarrow \mathbf{A}\mathbf{x} = \mathbf{0}$.

When $\mathbf{A}$ is invertible (i.e. $\det(\mathbf{A}) \neq 0$), the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ has one fixed point $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Classification of Equilibrium at (0,0)

- Distinct real eigenvalues: $\lambda_1 \neq \lambda_2$
  - $\lambda_1 > 0$, $\lambda_2 > 0 \Longrightarrow$ **source point, unstable**
  - $\lambda_1 \cdot \lambda_2 < 0 \Longrightarrow$ **saddle, unstable**
  - $\lambda_1 < 0$, $\lambda_2 < 0 \Longrightarrow$**sink node, asymptotically stable**
- Complex conjugate eigenvalues: $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.
  - $\alpha = 0 \Longrightarrow$ **center, stable (but not asymptotically)**
  - $\alpha < 0 \Longrightarrow$ **spiral point, asymptotically stable**
  - $\alpha > 0 \Longrightarrow$ **spiral point, unstable**
- Repeated real eigenvalue: $\lambda = \lambda_1 = \lambda_2$.
  If you have two eigenvectors
  - $\lambda > 0 \Longrightarrow$ **proper node, unstable**
  - $\lambda < 0 \Longrightarrow$ **proper node, asymptotically stable**

  otherwise
  - $\lambda > 0 \Longrightarrow$ **improper node, unstable**
  - $\lambda < 0 \Longrightarrow$ **improper node, asymptotically stable**