M251 Notes on Second Order ODE

Theorems:

Superposition principal: If $y_1(t)$ and $y_2(t)$ are solution of a second order homogeneous ODE

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0, \tag{1}$$

Then $\forall C_1, C_2 \in \mathbb{R}$, $y = C_1y_1 + C_2y_2$ is also a solution.

Existence & Uniqueness: Let us define the following second order linear ODE

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), \\ y(t_0) = y_0, \\ y'(t_0) = \bar{y}_0. \end{cases} \tag{2}$$

If the functions p, q, and g are continuous on the interval I: $\alpha < t < \beta$, containing the point t_0. Then there exists a unique solution of the ODE that is defined on I.

Wronskian: Let $f(t)$ and $g(t)$ be two functions. Then

$$W(f, g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g.$$

- $W(f, g) \equiv 0 \implies f$ and g are linearly dependent.
- Otherwise $\implies f$ and g are linearly independent.

Abel's identity: Let y_1 and y_2 be solution of

$$y'' + p(t)y' + q(t)y = 0, \tag{3}$$

Then

$$W(y_1, y_2) = C \cdot \exp\left(-\int p(t)dt\right).$$

General Solutions: Consider the following equation

$$y'' + p(t)y' + q(t)y = 0, \tag{4}$$

If y_1 and y_2 are both solution and $W(y_1, y_2)(t) \neq 0$ then the general solution of the ODE is

$$y = C_1y_1 + C_2y_2$$

Homogeneous second order ODE with constant coefficients:

The ODE is as follows

$$\boxed{ay'' + by' + cy = 0}.$$

The characteristic polynomial is ($y'' \to r^2$, $y' \to r$, $y \to 1$)

$$ar^2 + br + c = 0$$

Case 1: two real distinct roots $r_1 \neq r_2$.
Then the general solution is

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Case 2: two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$.
Then the general solution is

$$y = C_1 e^{\alpha t} \cos(\beta t) + C_2 e^{\alpha t} \sin(\beta t)$$

Case 3: one roots $r_1 = r_2 = \bar{r}$.
Then the general solution is

$$y = C_1 e^{\bar{r}t} + C_2 t e^{\bar{r}t} = e^{\bar{r}t} \left(C_1 + c_2 \cdot t\right)$$

Find a second solution (reduction of order)

You have the equation

$$y'' + p(t)y' + q(t)y = 0, \tag{5}$$

and assume you know one solution y_1. To find a second solution you define $\boxed{y_2 = v(t)y_1}$ and you obtain the following

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1(v' y_1 + v y_1') - v y_1 y_1' = v' y_1^2,$$

and from Abel's identity with $C = 1$ we have

$$W(y_1, y_2) = \exp\left(-\int p(t)dt\right).$$

Thus you solve for v

$$v' y_1^2 = \exp\left(-\int p(t)dt\right)$$