M251 Notes on Non-homogeneous Second Order ODE

Method/theorem

The goal is to solve

$$y'' + p(t)y' + q(t)y = g(t) \tag{N}$$

Steps

1. Solve the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{H}$$

   i.e finding y_1 and y_2, two linearly independent solutions. The we define

$$y_H = C_1 y_1 + C_2 y_2.$$

2. Find a particular solution Y of the non-homogeneous equation (N) (using the method of undetermined coefficients)

3. The general solution of (N) is

$$y = y_h + Y = C_1 y_1 + C_2 y_2 + Y$$

Method of undetermined coefficients

Now, we only consider the following ODE with constant coefficients

$$\boxed{ay'' + by' + cy = g(t)}$$

Let r_1 and r_2 be the roots of the characteristic equation of the homogeneous problem.

Case 1: $g(t) = \gamma e^{\alpha t}$

case	form of the particular solution Y
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$Y = Ae^{\alpha t}$
$r_1 = \alpha$ and $r_2 \neq \alpha$	$Y = Ate^{\alpha t}$
$r_1 = r_2 = \alpha$	$Y = At^2 e^{\alpha t}$

Case 2: $g(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$

case	form of the particular solution Y
$c \neq 0$	$Y = A_n t^n + \cdots + A_1 t + A_0$
$c = 0$ and $b \neq 0$	$Y = t\left(A_n t^n + \cdots + A_1 t + A_0\right)$
$c = b = 0$	$Y = t^2\left(A_n t^n + \cdots + A_1 t + A_0\right)$

Case 3: $g(t) = \gamma\cos(\beta t) + \delta\sin(\beta t)$

case	form of the particular solution Y
$r_{1,2} \neq \pm\beta i$	$Y = A\cos(\beta t) + B\sin(\beta t)$
$r_{1,2} = \pm\beta i$	$Y = t\,(A\cos(\beta t) + B\sin(\beta t))$

Case 4: $g(t) = e^{\alpha t}\left[\alpha_n t^n + \cdots + \alpha_1 t + \alpha_0\right]$

case	form of the particular solution Y
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$Y = e^{\alpha t}\left[A_n t^n + \cdots + A_1 t + A_0\right]$
$r_1 = \alpha$ and $r_2 \neq \alpha$	$Y = te^{\alpha t}\left[A_n t^n + \cdots + A_1 t + A_0\right]$
$r_1 = \alpha$ and $r_2 = \alpha$	$Y = t^2 e^{\alpha t}\left[A_n t^n + \cdots + A_1 t + A_0\right]$

Case 5: $g(t) = e^{\alpha t}\left[\gamma\cos(\beta t) + \delta\sin(\beta t)\right]$

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm \beta i$	$Y = e^{\alpha t}\left[A\cos(\beta t) + B\sin(\beta t)\right]$
$r_{1,2} = \alpha \pm \beta i$	$Y = te^{\alpha t}\left[A\cos(\beta t) + B\sin(\beta t)\right]$

Case 6: $g(t) = e^{\alpha t}\left[\gamma\cos(\beta t) + \delta\sin(\beta t)\right](\eta_n t^n + \cdots + \eta_1 t + \eta_0)$
or $= e^{\alpha t}\left[(\eta_n t^n + \cdots + \eta_1 t + \eta_0)\cos(\beta t) + (\mu_n t^n + \cdots + \mu_1 t + \mu_0)\sin(\beta t)\right]$

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm \beta i$	$Y = e^{\alpha t}\left[(A_n t^n + \cdots + A_1 t + A_0)\cos(\beta t) + (B_n t^n + \cdots + B_1 t + B_0)\sin(\beta t)\right]$
$r_{1,2} = \alpha \pm \beta i$	$Y = te^{\alpha t}\left[(A_n t^n + \cdots + A_1 t + A_0)\cos(\beta t) + (B_n t^n + \cdots + B_1 t + B_0)\sin(\beta t)\right]$

Case 7: $g(t)$ is a sum of the previous cases, i.e.

$$g(t) = g_1(t) + \cdots + g_n(t).$$

You find a particular solution Y_i for each $g_i(t)$ term as if it were the only term in $g(t)$. Then

$$Y = Y_1(t) + \cdots + Y_n(t).$$

Rules

1. Take Y as the same form as $g(t)$.
2. If $g(t)$ is a solution to the homogeneous equation then multiply by t.
3. If $t \cdot g(t)$ is still solution of the homogeneous equation then multiply by t^2