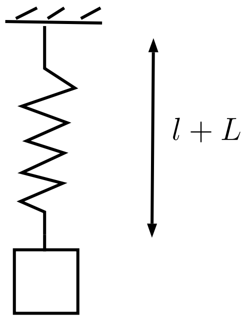
M251 Notes on Spring Mass System

Mass Spring System



$$mu'' = ma = \sum f = mg - k(L+u) - \gamma u' + F$$

Since $mg = kL$ we have

$$\boxed{mu'' + \gamma u' + ku = F(t)}$$

with

- $u(t)$ = displacement of the mass relative to its equilibrium position.
- m = mass $(m > 0)$
- γ = damping constant $(\gamma \geq 0)$
- k = spring (Hooke's) constant $(k > 0)$
- g = gravitational constant
- L = elongation of the spring caused by the weight
- $F(t)$ = Externally applied forcing function, if any
- $u(t_0)$ = initial displacement of the mass
- $u(t_0)$ = initial velocity of the mass

Unforced Mass-Spring System ($F(t) = 0$)

Undamped free vibration ($\gamma = 0$)

$$\boxed{mu'' + ku = 0},\ m > 0,\ k > 0$$

Solve it

$$mr^2 + k = 0,\ \ r^2 = -\frac{k}{m},\ \ r_{1,2} = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0,\ \text{where } w_0 = \sqrt{\frac{k}{m}}$$

The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

with

- **Frequency**: $\omega_0 = \sqrt{\frac{k}{m}}$
- **Period**: $T = \frac{2\pi}{\omega_0}$
- **Amplitude**: $R = \sqrt{C_1^2 + C_2^2}$

Damped free vibration ($\gamma \neq 0$)

$$\boxed{mu'' + \gamma u' + ku = 0},\ m > 0,\ \gamma > 0,\ k > 0$$

$\Delta = \gamma^2 - 4km$, the roots are $r_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}$

- if $\Delta > 0$ then **Overdamped** $\Longrightarrow u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$.
- it $\Delta = 0$ then **Critically damped** $\Longrightarrow u(t) = C_1 e^{rt} + C_2 t e^{rt}$.
- it $\Delta < 0$ then **Underdamped**. $\Longrightarrow u(t) = e^{-\lambda t}\left[C_1 \cos(\mu t) + C_2 \sin(\mu t)\right]$.

Undamped with forced vibration

$$\boxed{mu'' + ku = F_0 \cos(\omega t)} \text{ or } \boxed{mu'' + ku = F_0 \sin(\omega t)},$$

with $m > 0$ and $k > 0$.

<u>Case 1</u> $\omega \neq \omega_0$
The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + A \cos(\omega t) + B \sin(\omega t).$$

<u>Case 2</u> $\omega = \omega_0$
The solution is

$$u(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + At \cos(\omega_0 t) + Bt \sin(\omega_0 t).$$

We say that the system is in **resonance**.

$$\boxed{\omega = \omega_0 \Longrightarrow \text{Resonance}}$$