M251 Notes on System of ODE

Matrices

A 2×2 matrix is

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A vector is

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$$

The identity is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For any matrix $\mathbf{A}$ we have

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

For any vector $\mathbf{v}$ we have

$$\mathbf{Iv} = \mathbf{v}$$

Matrix Multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} ax + bz & ay + bu \\ cx + dz & cy + du \end{bmatrix}$$

Vector Multiplication (usually $\mathbf{AB} \neq \mathbf{BA}$)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + cy \end{bmatrix}$$

Determinant

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cd$$

Inverse of A exists if and only if $\det(A) \neq 0$

$$\mathbf{A^{-1}A} = \mathbf{AA^{-1}} = \mathbf{I}$$

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Longrightarrow \mathbf{A^{-1}} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition. λ is an eigenvalue of $\mathbf{A}$ and $\mathbf{v} \neq 0$ is an eigenvector associated to λ if

$$\mathbf{Av} = \lambda \mathbf{v}$$

Theorem. The eigenvalues of $\mathbf{A}$ are the roots of

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \mathbf{0}$$

The eigenvector $\mathbf{v}$ associated to λ solves

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = \mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

System of ODE

We have

$$\begin{cases} x_1' = ax_1 + bx_2 \\ x_2' = cx_1 + dx_2 \end{cases} \iff \mathbf{x}' = \mathbf{A}\mathbf{x} \quad \text{where } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solving a system

1. Find the eigenvalues λ_1, λ_2
2. Find the eigenvectors $\mathbf{v_1}$, $\mathbf{v_2}$
3. Find two linearly independent solutions, and write the general solution

The general solutions are

- **Distinct real eigenvalues:** $\lambda_1 \neq \lambda_2$, then the general solution is

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- **Complex conjugate eigenvalues:** $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$.
  Then the eigenvectors are $\mathbf{v}_1 = \mathbf{a} + i\mathbf{b}$ and $\mathbf{v}_2 = \mathbf{a} - i\mathbf{b}$, and the general solution is

$$\mathbf{x} = C_1 e^{\alpha t}\left[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}\right] + C_2 e^{\alpha t}\left[\cos(\beta t)\mathbf{b} + \sin(\beta t)\mathbf{a}\right]$$

- **Repeated real eigenvalue:** $\lambda = \lambda_1 = \lambda_2$.

If you can find two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{x} = C_1 e^{\lambda t}\mathbf{v}_1 + C_2 e^{\lambda t}\mathbf{v}_2$$

If not, take $\mathbf{v}$ eigenvector associated to λ

$$\mathbf{x} = C_1 e^{\lambda t}\mathbf{v} + C_2\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\boldsymbol{\eta}\right)$$

where $\boldsymbol{\eta}$ solves $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \mathbf{v}$